# An Automata-Based Approach to Games with $\omega$-Automatic Preferences

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# Subject

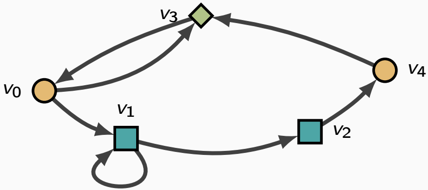
Joint work with Véronique Bruyère, Emmanuel Filiot, Jean-François Raskin

**Reactive systems**: **continuous interactions** between multiple **independent agents** with their own interests.

$\rightsquigarrow$ Ensure some specification under **rational** assumptions of agents.

# Game played on graphs

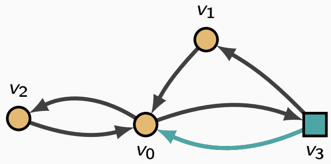
⇝ Model interactions with **games played on graphs**.




$$\left.\begin{array}{l}\text{Directed graph: } (V, E) \\ \text{Set of players: } \mathcal{P} = \{1, \dots, n\} \\ \text{Partition of } V\text{: } (V_i)_{i \in \mathcal{P}}\end{array}\right\} \text{Arena } A = (V, E, \mathcal{P}, (V_i)_{\in \mathcal{P}})$$

- **Play**: $\pi \in \text{Plays} \subseteq V^\omega$ consistent with $E$, history: $h \in V^*$,
- **Strategy** for $i \in \mathcal{P}$: function $\sigma_i : V^* V_i \to V$, $hv \mapsto \sigma_i(hv)$.

# Games played on graphs



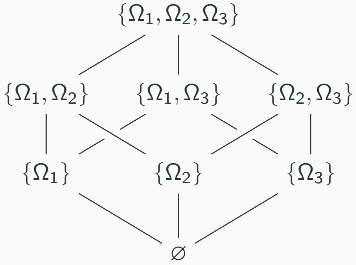

Can player ∘ ensure the objective of **visiting** $v_1$ from $v_0$?

Classical setting: **objectives** $\Omega_i : V^\omega \to \mathbb{Q}$, for each player $i \in \mathcal{P}$ (e.g., $\Omega_i : V^\omega \to \{0,1\}$).

Broader setting: **preference relations** $\ltimes_i \subseteq V^\omega \times V^\omega$ to compare plays:

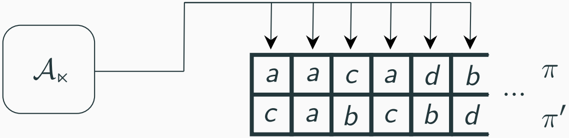
$\rightsquigarrow \pi \ltimes_i \pi'$ if $\pi'$ is **preferred** to $\pi$,

e.g., $\pi \ltimes_i \pi'$ if $\Omega_i(\pi) < \Omega_i(\pi')$.



# $\omega$-Automatic Relations

$\rightsquigarrow$ Define $\ltimes$ with a **deterministic parity automaton** (DPW) $\mathcal{A}_\ltimes$ on the alphabet $V \times V$ that **synchronously** reads two $\omega$-words.



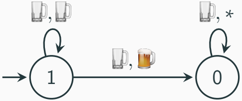

$\mathcal{L}(\mathcal{A}) \subseteq (V \times V)^\omega$ can be seen as a binary relation: **$\omega$-automatic relation**!

**Examples (Alphabet $\Sigma = \{a, b\}$)**

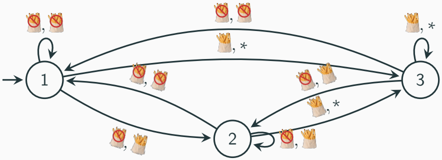
- $\{(x, y) \mid \text{first}_a(x) \leq \text{first}_a(y)\}$ is $\omega$-automatic,
- $\{(x, y) \mid \text{for each prefix } u \text{ of } x,\ u[a \leftarrow aa] \text{ is a prefix of } y\}$ is **not** $\omega$-automatic ($\rightsquigarrow$ need desynchronization).

$x \ltimes y$ **iff** $\Omega(x) < \Omega(y)$ (Boolean: $\Omega(x) = 0$ and $\Omega(y) = 1$)

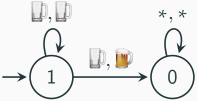
Reach(🍺)

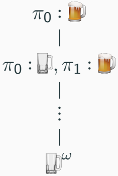



Buchi(🍟)

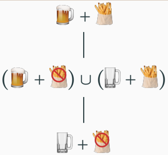



Reach(🍺) "as soon as possible"

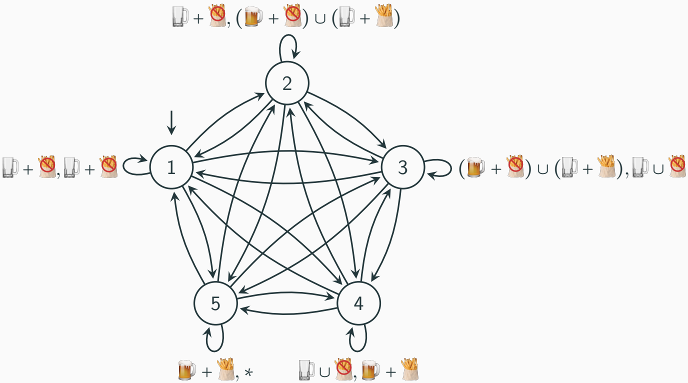


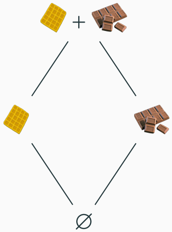
# Expressivity – Combinations of Classical Objectives



**Counting order** with two Büchi objectives (🍺 and 🍟).



**Subset order** with two reachability objectives (, ): 12 states.




**Theorem [Bansal, Chaudhuri, Vardi 2022]**

- Limsup objectives: $\omega$-automatic.
- Discounted-Sum objectives: $\omega$-automatic iff the discount-factor is an integer $d \in \mathbb{N}$.
- Mean-Payoff objectives: **not** $\omega$-automatic.

# Zero-Sum Games

Zero-Sum: player 1 VS player 2.

**Threshold problem**

Given a lasso $\pi \in \mathsf{Plays}(v_0)$, does player 1 have a **winning strategy for the objective** $\Omega = \{\rho \in V^\omega \mid \pi \ltimes \rho\}$?

**Study values**: $\mathsf{Val}_1(v) = \{\pi \in \mathsf{Plays}(v) \mid \exists \sigma_1 \forall \sigma_2,\ \pi \ltimes \langle \sigma_1, \sigma_2 \rangle_v\}$

$\leadsto$ The threshold problem amounts to check $\pi \in \mathsf{Val}_1(v_0)$.

**Optimal strategy**: $\sigma_1^*$ such that $\forall \pi \in \mathsf{Val}_1(v)$, $\forall \sigma_2$, $\pi \ltimes \langle \sigma_1^*, \sigma_2 \rangle_v$.

**Lemma**

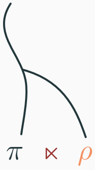
$\mathsf{Val}_1(v)$ is accepted by a polysize **alternating** parity automaton (APW).

# Recognize $\mathrm{Val}_1(v) = \{\pi \in \mathrm{Plays}(v) \mid \exists\sigma_1\forall\sigma_2,\ \pi \ltimes \langle\sigma_1,\sigma_2\rangle_v\}$

$\pi$: Input

$\rho$, player 1: $\exists$

$\rho$, player 2: $\forall$




| Input $\pi : \pi_0$ |
|---|
| Dev $\rho : \rho_0$ |
| $q_0$ |

$(\pi_0 = \rho_0 = v_0)$

$\bigvee_{(\rho_0,\rho_1)\in E} \longrightarrow$ if $\rho_0 \in V_1$: Player 1 chooses one $\rho_1$ for $\rho$

| Input $\pi : \pi_1$ |
|---|
| $\rho_1$ |
| $q_1 = \delta(q_0, (\pi_0, \rho_0))$ |

$\bigwedge_{(\rho_1,\rho_2)\in E} \longrightarrow$ if $\rho_1 \in V_2$: Player 2 chooses one $\rho_2$ for $\rho$

| Input $\pi : \pi_2$ |
|---|
| $\rho_2$ |
| $q_2 = \delta(q_1, (\pi_1, \rho_1))$ |

# Zero-Sum Games

Recall $\mathsf{Val}_1(v) = \{\pi \in \mathsf{Plays}(v) \mid \exists \sigma_1 \forall \sigma_2,\ \pi \ltimes \langle \sigma_1, \sigma_2 \rangle_v\}$.

| Given | Question | Equivalence | Complexity |
|---|---|---|---|
| $\pi$ | Threshold prob. with $\pi$? | $\pi \in \mathsf{Val}_1(v_0)$? | Parity-complete |
| $\pi, \sigma_1$ | Threshold prob. with $\pi$, $\sigma_1$? | — | NL-complete |
| / | $\exists \pi$, threshold prob. with $\pi$? | $\mathsf{Val}_1(v_0) \neq \emptyset$? | PSPACE-complete |
| $\sigma_1$ | Is $\sigma_1$ optimal? | $\neg(\mathsf{Val}_1(v_0) \cap \sigma_1 \cap \ldots = \emptyset)$? | PSPACE-complete |
| / | $\exists \sigma_1$ optimal? | Win. strat., obj. $\mathsf{Val}_1(v_0)$? | 2EXPTIME, PSPACE-h |

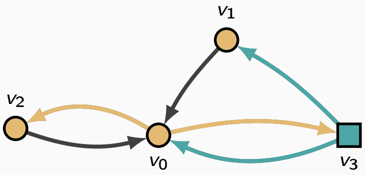
Let $\mathsf{Val}_{-1}(v) = \{\pi \in \mathsf{Plays}(v) \mid \exists \sigma_2 \forall \sigma_1,\ \pi \not\ltimes \langle \sigma_1, \sigma_2 \rangle_v\}$

$\rightsquigarrow \mathsf{Val}_1(v) = \mathsf{Plays}(v) \setminus \mathsf{Val}_2(v)$

# Beyond Zero-Sum: Non-Zero-Sum Games

A **Nash Equilibrium** (NE) is a strategy profile $\sigma = (\sigma_i)_{i \in \mathcal{P}}$ from which no player has the incentive to **unilaterally deviate**, i.e.,

$$\forall i \in \mathcal{P}, \forall \tau_i \text{ strategy of player } i,\ \langle \sigma \rangle_{v_0} \not\prec_i \langle \sigma_{-i}, \tau_i \rangle_{v_0}.$$




- player ○ wants to visit $v_1$,
- player □ wants to visit $v_2$ inf. often.

**Do there exist NEs from $v_0$?**

1) $\sigma_\square(hv_3) = v_0$, $\sigma_\circ(hv_0) = v_3$.
2) $\sigma'_\square(hv_3) = v_1$, $\sigma'_\circ(hv_0) = v_2$ if $v_1$ in $h$, else $v_3$.

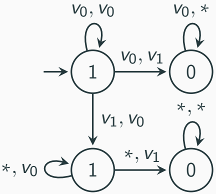
NE 2) is **strictly better** than NE 1) for both players.

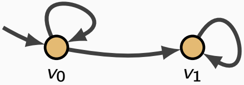
# NE Existence

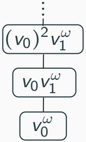
Does there **always** exist a Nash Equilibrium?

We can encode **“reachability as late as possible”**:

$\pi \ltimes \pi'$ if $\pi'$ visits $v_1$ after $\pi$, or if $\pi$ never visits $v_1$.



# Decision Problems

**NE Threshold problem**

Given lassos $(\rho_i)_{i \in \mathcal{P}}$, does there exist an NE $\bar{\sigma}$ such that $\rho_i \ltimes_i \langle \bar{\sigma} \rangle_{v_0}$ for all $i \in \mathcal{P}$?

In general: $\omega$-**regular constraint** $C \subseteq V^\omega$ given as an **APW**.

**Theorem**

The NE threshold problem is PSPACE-complete.

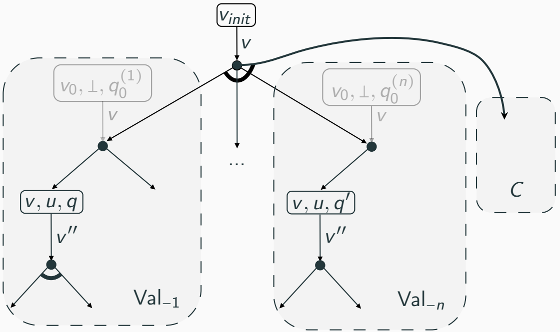
Improve the previous 2EXPTIME bound [Bruyère, G., Raskin 2025].

**Theorem**

Deciding the existence of a Pareto-Optimal (maximal) NE outcome is in EXPSPACE and PSPACE-hard.

# NE Existence Problem in PSPACE

- Using value $\mathsf{Val}_{-i}(v) = \{\pi \in \mathsf{Plays}(v) \mid \exists \sigma_{-i} \forall \sigma_i, \pi \not\prec_i \langle \bar{\sigma} \rangle_v\}$,
- We have $\mathsf{NE_{out}}(v) = \bigcap_{i \in \mathcal{P}} \mathsf{Val}_{-i}(v)$,
- Compute $C \cap \mathsf{NE_{out}}(v)$ with **one universal transition**




$\rightsquigarrow$ Polynomial size, **emptiness** algorithm in PSPACE.

# Rational Synthesis (Stackelberg Game)

A special player, player 0, declares $\sigma_0$ first and is **forbidden** to deviate.

Given a threshold lasso $\pi \in \mathrm{Plays}(v)$,

**Cooperative NE rational synthesis (CRS)**:

$$\exists \sigma_0, \exists \sigma_0\text{-fixed NE } \bar{\sigma}, \quad \pi \ltimes_0 \langle \bar{\sigma} \rangle_v .$$

**Non-cooperative NE rational synthesis (NCRS)**:

$$\exists \sigma_0, \forall \sigma_0\text{-fixed NE } \bar{\sigma}, \quad \pi \ltimes_0 \langle \bar{\sigma} \rangle_v .$$

**Theorem**

The CRS problem is PSPACE-complete.

The NCRS problem is undecidable. (Reduction from the $\omega$-$PCP$ problem)

# Future/Ongoing Work

⇝ Close the **complexity gaps** for optimal strategies and Pareto-Optimal NE (existential projection for APW requires non-determinism).

⇝ Main future work: extend to **subgame perfect equilibria (SPE)**.

- There are games with the existence of NEs but with **no SPE**.
- Study **one-deviation immune** profiles, **weak SPEs**,
- **Characterize** or decide the **existence** of (weak) SPE outcomes,
- The **SPE existence** problem for **one-player games** is PSPACE-complete.

Thank you! Questions?

# Undecidability of Rational Synthesis

From *ω-PCP*: Let $\varphi_L, \varphi_R : A \to B^*$, find $a \in A^\omega$ such that $\varphi_L(a) = \varphi_R(a)$.